Homefun #1 (15 points)

Due: February 1st, 2012

Problem 1. (1 point) Here is a formal language consisting of three strings.

John ate a sandwich
Jim ate a sandwich
John and Jim ate a sandwich

Illustrate how “symbol” in the formal language sense does not have to correspond exactly to “character in the English alphabet”: give three different alphabets that could each be used to write out a language like this. That is, above, you see some marks on the page representing strings; but what you think of as “symbols” is up to you—so define three different sets of symbols that would allow you to print these marks on the page as strings. For each of the three alphabets, state the length of the corresponding three strings.

Problem 2. (2 points) Spell out any missing links in the following argument, and correct any important terminological mistakes, explaining carefully how the meaning of the corrected version differs from the original:

In English, we can always embed sentences, as in John thought that Mary said that Bill left, John thought that Mary said that Bill claimed that John left, and so on. Thus English sentences can be infinitely long, and so, if we see English as a formal language, it would be an infinite language.

Problem 3. (6 points) Formally, a sequence of real numbers is a partial function \( f : \mathbb{N}^+ \rightarrow \mathbb{R} \) returning the \( i \)th element of the sequence. For example, if we were asked to define the sequence \( \langle 1, 2, 4, 8 \rangle \), we might give the following function:

\[
f(i) = \begin{cases} 
2^{i-1} & \text{if } 1 \leq i \leq 4 \\ 
\text{undefined} & \text{if } i > 4 
\end{cases}
\]

We can think of \( f \) itself as “the sequence,” rather than “a function defining the sequence”; in other words, we can think of \( \langle 1, 2, 4, 8 \rangle \) and the functional definition given above as simply two different ways of writing \( f \). Here, we stipulate that if a sequence is defined for \( i > 1 \), then it must also be defined for \( i - 1 \) (i.e., there can be no gaps in the sequence, and the sequence must start at 1).

The \( k \)th partial sum function is given by a mapping taking a sequence \( f \) to a real number, \( S_k : (\mathbb{N}^+ \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \). It is defined recursively as follows:

\[
S_k(f) = \begin{cases} 
S_{k-1}(f) + f(k) & \text{if } f(k) \text{ is defined} \\
0 & \text{if } k = 0 \\
\text{undefined} & \text{otherwise if } f(k) \text{ is not defined}
\end{cases}
\]
Finally, the *summation function* for a sequence, $S : (\mathbb{N}^+ \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$, can be defined as $S(f) = S_n(f)$, where $n$ is the largest integer for which the partial sum function is defined. We write $S(f)$ using the familiar summation notation: $\sum_{i=1}^n f(i) = S(f)$, where $S(f) = S_n(f)$.

Using only these definitions and basic arithmetic, prove the following familiar summation equalities:

(a) (3 out of 6 points) $\sum_{i=1}^n c \cdot (f(i) + g(i)) = c \sum_{i=1}^n f(i) + c \sum_{i=1}^n g(i)$

(b) (3 out of 6 points) $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$

*Clarification 1.* You will need to use induction. Make sure to state the inductive hypothesis explicitly and mark clearly the steps where you use it.

*Clarification 2.* This problem is to warm you up to doing proof by induction. However, it is different from the problems we will do for the rest of the class. The proofs we usually do in formal language theory use a lot of words to make each precise each step in the reasoning; they use words to avoid introducing a lot of extra notation. In other words, the proofs we do about formal languages/automata are not *themselves* formal proofs. However, for these problems, you *have* notation (i.e., the usual notation for arithmetic), and you should use it. Do not try and explain the steps in words!

**Problem 4.** (6 points) (Linz 2012) A *rational number* is a number that can be expressed as the ratio of two integers $n$ and $m$, $\frac{n}{m}$, where $m \neq 0$. A real number that is not rational is said to be irrational. Prove or disprove the following:

(a) (3 out of 6 points) The sum of a rational and an irrational number is always irrational. [You may use the fact that $\mathbb{Z}$ is closed under arithmetic operations.]

(b) (3 out of 6 points) The sum of two positive irrational numbers is always irrational. [Use the result from (a). You do *not* need to pretend you do not know any examples of irrational numbers.]

**References**